Q ⇒ If $u = x^2 \tan^{-1}\frac{y}{x} - y^2 \tan^{-1}\frac{x}{y}$.

Show that $\frac{\delta^2 u}{\delta x \delta y} = \frac{x^2 - y^2}{x^2 + y^2}$

and $\frac{\delta^2 u}{\delta x \delta y} = \frac{\delta^2 u}{\delta y \delta x}$.

Sol^n ⇒ $u = x^2 \tan^{-1}\frac{y}{x} - y^2 \tan^{-1}\frac{x}{y}$.

$$\Rightarrow \frac{\delta u}{\delta y} = x^2 \cdot \frac{1}{1+(y/x)^2} \cdot \frac{1}{x} - \left\{ 2y \cdot \tan^{-1}\frac{x}{y} + y^2 \frac{1}{1+\left(\frac{x}{y}\right)^2}\left(-\frac{x}{y^2}\right)\right\}$$

$$\Rightarrow \frac{\delta u}{\delta y} = \frac{x^3}{x^2+y^2} - 2y \tan^{-1}\frac{x}{y} + \frac{xy^2}{x^2+y^2} = x - 2y \tan^{-1}\frac{x}{y}.$$

$$\Rightarrow \frac{\delta^2 u}{\delta x \delta y} = \frac{\delta}{\delta x}\left[x - 2y \tan^{-1}\frac{x}{y}\right\} = 1 - 2y \cdot \frac{1}{1+\left(\frac{x}{y}\right)^2} = 1 - \frac{2y^2}{x^2+y^2}$$

$$\Rightarrow \boxed{\frac{\delta^2 u}{\delta x \delta y} = \frac{x^2 - y^2}{x^2 + y^2}}$$

Similarly, $\frac{\delta u}{\delta x} = 2x \tan^{-1}\frac{y}{x} - y$ [solve it]

$$\Rightarrow \frac{\delta^2 u}{\delta y \delta x} = \frac{\delta}{\delta y}\left[2x \tan^{-1} x\frac{y}{x} - y\right\} = \frac{x^2 - y^2}{x^2 + y^2}$$ [verify again].